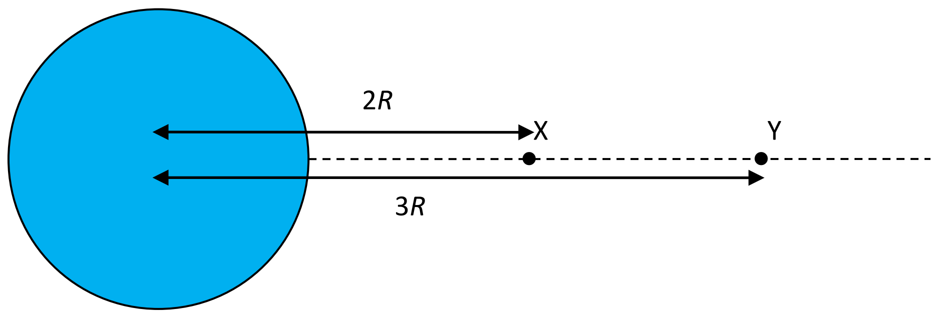
Problem of the week

Gravitational fields (HL)

(a) X and Y are two points on a radial line away from a spherical planet of mass 5.0×10^{25} kg and radius *R*. The planet has no atmosphere.

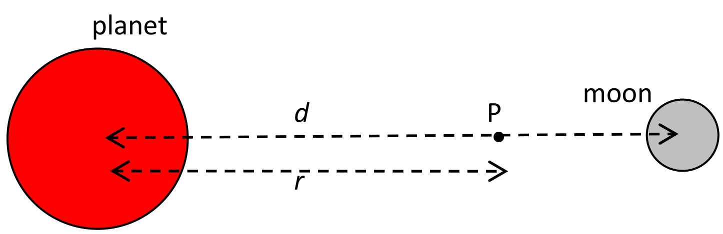


(i) Show that the difference in gravitational potential between points X and Y is $\frac{GM}{6R}$.

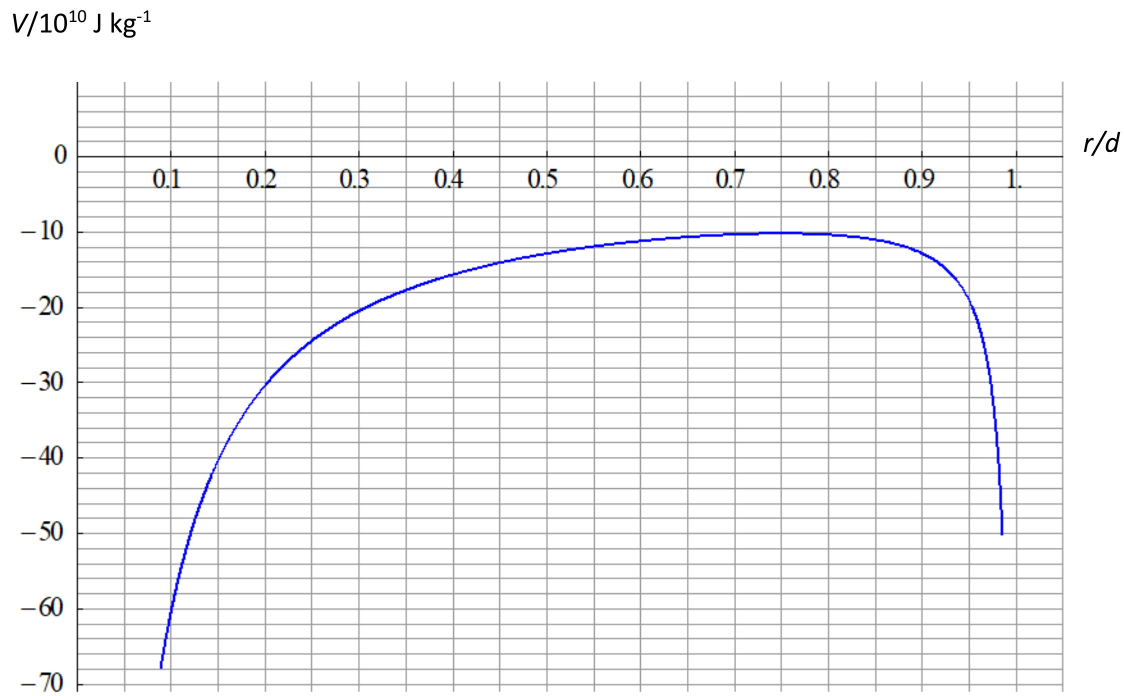
(ii) A rocket moves along the radial line. The rocket runs out of fuel at point X. At X the speed of the rocket is 8.0 km s^{-1}. When the rocket gets to point Y the speed becomes 6.0 km s^{-1}. Determine the radius of the planet.

(iii) Discuss whether the rocket will be able to escape the gravitational field of the planet.

(b) Point P is a distance *r* from the center of a planet. The center of a moon is a distance *d* from the center of the planet. The graph shows the variation with *r*/*d* of the gravitational potential *V* at P due to the planet and the moon..



The graph starts at the surface of the planet and ends at the surface of the moon.



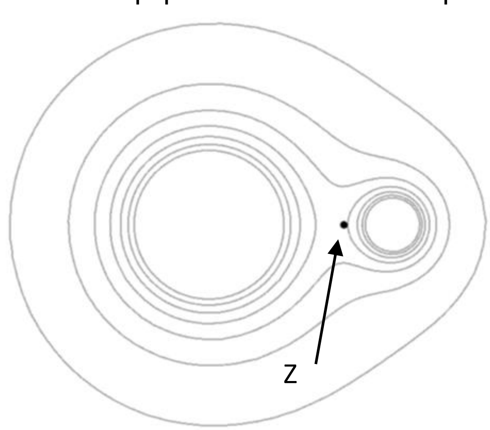
(i) Determine the ratio $\frac{M_{\text{planet}}}{M_{\text{moon}}}$ of the mass of the planet to the mass of the moon.

(ii) An amount of energy E is supplied to a projectile of mass 850 kg on the surface of the planet so that it reaches the surface of the moon.

Calculate the minimum value of E.

(iii) The energy in (ii) has been supplied to the projectile. Determine the speed of the projectile as it crashes on the surface of the moon.

(iv) The figure shows equipotential lines for the planet and the moon.



Discuss the significance of point Z.

(v) Suggest why the equipotential lines tend to become circular as they grow larger.

A satellite of mass m = 650 kg orbits the earth (mass $M = 6.0\times10^{24}$ kg) in a circular orbit at a height of 520 km above the earth's surface. The radius of earth is 6.37×10^{6} m.

(c) Determine, for this satellite,

(i) the orbital speed,

(ii) the orbital period.

(d) A small frictional force f acts on the satellite.

(i) Explain why the orbital radius of the satellite will decrease and the speed will increase.

(ii) During one revolution the orbit radius decreases from r to $r-\Delta r$. Show that the decrease in the total energy of the satellite is given approximately by $\Delta E \approx \dfrac{GMm}{2r^2}\Delta r$.

(iii) The height of the satellite decreases by 0.90 m after one revolution. Calculate the loss of energy after one revolution at a height of 520 km.

(iv) Estimate the average rate at which energy was dissipated during the one revolution.

(v) Estimate the magnitude of f. You may assume that f acts opposite to the velocity.

(vi) Comment on the answer to (v).

Answers

(a)

(i)

$$\begin{aligned}\Delta V &= V_Y - V_X \\ &= -\frac{GM}{3R} - (-\frac{GM}{2R}) \\ &= \frac{GM}{6R}\end{aligned}$$

(ii) Apply conservation of energy between X and Y: $\left(\frac{1}{2}mv^2 + mV\right)_X = \left(\frac{1}{2}mv^2 + mV\right)_Y$ so that $\frac{1}{2}v_Y{}^2 - \frac{1}{2}v_X{}^2 = V_X - V_Y$

OR

$-m\Delta V = \Delta E_K$ (work is done by gravity hence the minus sign)

$$-m\frac{GM}{6R} = \frac{1}{2}m(v_Y^2 - v_X^2)$$

$$R = \frac{GM}{3(v_X^2 - v_Y^2)}$$

$$R = \frac{6.67\times10^{-11}\times5.0\times10^{25}}{3\left((8.0\times10^3)^2 - (6.0\times10^3)^2\right)} = 3.97\times10^7 \approx 4.0\times10^7 \text{ m}$$

(iii) The escape speed at X is $v = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2\times6.67\times10^{-11}\times5.0\times10^{25}}{2\times3.97\times10^7}} = 9.17\times10^3 \text{ m s}^{-1}$. At X the rocket has a speed less than the escape speed and so will not escape.

(iv) The equipotential lines wrap around each body but as they grow larger they wrap around both bodies. Z is the point where this transition takes place. We can think of Z as belonging to both an equipotential wrapping around the planet and an equipotential wrapping around the moon. This means that the potential near Z is not changing along the line joining the bodies. Then the gradient of the potential at Z must be zero. So, Z is where the gravitational field strength is zero.

(v) From far away the planet and the moon appear as one mass and the equipotential surfaces of a single mass are spheres, and so, circles on a 2-d representation.

(b)

(i) At $\frac{r}{d} = 0.75$, g = 0 and so $\frac{GM_{\text{planet}}}{(0.75d)^2} = \frac{GM_{\text{moon}}}{(0.25d)^2}$, hence $\frac{M_{\text{planet}}}{M_{\text{moon}}} = \frac{0.75^2}{0.25^2} = 9$.

(ii) The energy must be enough to bring the probe to $\frac{r}{d}=0.75$. The energy is then $m\Delta V = 850\times(-10\times10^{10}-(-68\times10^{10}))=4.9\times10^{14}$ J.

(iii) By conservation of energy: $0+m\times(-10\times10^{10})=\frac{1}{2}mv^2+m\times(-50\times10^{10})$. Hence $v^2=80\times10^{10}$ giving $v=8.9\times10^3\,\text{m s}^{-1}$.

(c)

(i) $v=\sqrt{\frac{GM}{r}}=\sqrt{\frac{6.67\times10^{-11}\times6.0\times10^{24}}{(6.37+0.520)\times10^6}}=7.621\times10^3\approx7.6\times10^3\,\text{m s}^{-1}$.

(ii) $T=\frac{2\pi r}{v}=\frac{2\pi(6.37\times10^6+520\times10^3)}{7.621\times10^3}=5.25\times10^3$ s

(d)

(i) The total energy decreases so the radius gets smaller to make the total energy more negative, i.e., smaller. From $v=\sqrt{\frac{GM}{r}}$, speed increases since radius decreases.

(ii) We calculate the difference in total energy in the two orbits:

$$
\begin{aligned}
\Delta E &= -\frac{GMm}{2}\left(\frac{1}{r-\Delta r}-\frac{1}{r}\right)\\
&= -\frac{GMm}{2}\left(\frac{r-(r-\Delta r)}{r(r-\Delta r)}\right)\\
&= -\frac{GMm}{2}\left(\frac{\Delta r}{r(r-\Delta r)}\right)\\
&\approx -\frac{GMm}{2}\left(\frac{\Delta r}{r^2}\right) \qquad \text{since } r(r-\Delta r)\approx r^2
\end{aligned}
$$

The decrease in energy is then $\Delta E\approx\frac{GMm}{2r^2}\Delta r$.

(iii) $\Delta E \approx \frac{6.67\times10^{-11}\times6.0\times10^{24}\times650}{2\times(6.37+0.520)\times10^6)^2}\times0.90=2.467\times10^3\approx2.5\times10^3$ J.

(iv) $\bar{P}=\frac{2.467\times10^3}{5.25\times10^3}=0.4699\approx0.47$ W.

(v) $f\times2\pi r=\Delta E\Rightarrow f=\frac{\Delta E}{2\pi r}=\frac{2.467\times10^3}{2\pi(6.37\times10^6+520\times10^3)}\approx6\times10^{-5}$ N ***OR***

$\bar{P}=fv\Rightarrow f=\frac{0.4699}{7.621\times10^3}\approx6\times10^{-5}$ N.

(vi) The force is very small. This is mainly because the density of air at a height of 520 km is very small and because the surface area of a 650 kg satellite cannot be very large.